Proof of Prop 2.1

Now for $L = \mathbb{Z}\tau + \mathbb{Z}$ the proof of the lemma shows that

$$|G_k(\tau)| \leq \left|\sum_{m,n}{}' \frac{1}{(m\tau+n)^k}\right| \leq \sum_{m,n}{}' \frac{1}{|m\tau+n|^k}$$

$$\leq C^{-k/2} \sum{}' (m^2+n^2)^{-k/2} \quad \text{with } C = \frac{y^2}{x^2+y^2+1}$$

$\sum_{m,n}' (m^2+n^2)^{-k/2}$ is a convergent series indep. of $\tau$.

As for $C^{-k/2} = \left(\frac{y^2}{x^2+y^2+1}\right)^{-k/2}$ note that if

$\tau \in K$ a compact set then

$\inf_{\tau \in K} \operatorname{Im}\tau =: y_0$ is reached and is strictly positive

$\sup_{\tau \in K} (\operatorname{Re}\tau) =: x_0 < \infty$ and $\sup_{\tau \in K} \operatorname{Im}\tau =: y_1 < \infty$

So $C > \frac{y_0^2}{x_0^2+y_1^2+1} > 0$

Proving unif conv of $G_k(\tau)$ on compacta. ∎

Defn Normalized Eisenstein series is

$$E_k(z) := \frac{1}{2\zeta(k)} G_k(z)$$

Note $\lim_{\tau \to i\infty} E_k(z) = 1$

Remark For $k=2$ $\sum_{m,n}' \frac{1}{(m\tau+n)^2}$ does not

converge abs. Let

$$G_2(\tau) = \sum_{n \neq 0} \frac{1}{n^2} + \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau+n)^2}$$

One can show that

$$G_2\left(-\frac{1}{\tau}\right) = \tau^2 G_2(\tau) - \pi i \tau .$$

Hence $G_2$ is not modular.

(For a proof of this see J. Serre. A course in Arithmetic for example)

We might come back to this example

Next we'll look at the Fourier expansion of $G_k(\tau)$

<u>Prop 2.2</u> The Eisenstein series $G_k(\tau)$ ($k$ even, $k>2$) has the Fourier expansion

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} \sigma_{k-1}(n)\, q^n$$

$$(q = e^{2\pi i \tau}) \quad \text{where} \quad \sigma_s(n) := \sum_{d \mid n} d^s$$

<u>Proof</u> One can give several proofs of this expansion. We start with the most classical one even though it is not necessarily the most natural one.

It starts with the partial fraction expansion of the cotangent (see Ahlfors Complex Analysis p. 189)

$$\frac{1}{\tau} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\left(\frac{1}{\tau+n} - \frac{1}{n}\right) = \pi \cot \pi\tau$$

$$= \pi \frac{\cos \pi\tau}{\sin \pi\tau}$$

$$= \pi i \frac{e^{\pi i\tau} + e^{-\pi i \tau}}{e^{\pi i \tau} - e^{-\pi i \tau}}$$

$$= \pi i \frac{e^{2\pi i \tau} + 1}{e^{2\pi i \tau} - 1} = \pi i \left(1 - \frac{2}{1 - e^{2\pi i \tau}}\right)$$

$$= \pi i - 2\pi i \sum_{n=0}^{\infty} e^{2\pi i n \tau}$$

Differentiating both sides $k-1$ times wrt $\tau$ gives

$$(L)\quad \boxed{\sum_{n=-\infty}^{\infty} (\tau+n)^{-k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n \tau}}$$

This identity is called Lipschitz's formula

We'll use this as follows

$$G_k(\tau) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^k} + \sum_{\substack{m,n\in\mathbb{Z}\\ m\neq 0}} \frac{1}{(m\tau+n)^k}$$

$$= 2\zeta(k) + 2\sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$$

$$\overset{(L)}{=} 2\zeta(k) + 2\sum_{m=1}^{\infty} \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n (m\tau)}$$

$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \underbrace{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{k-1} e^{2\pi i (mn)\tau}}_{mn=N}$$

$$\sum_{N=1}^{\infty} \underbrace{\sum_{n|N} n^{k-1}}_{\shortparallel} e^{2\pi i N\tau}$$

$$= \sum_{N=1}^{\infty} \sigma_{k-1}(N)\, e^{2\pi i N\tau}.$$

where $\sigma_{k-1}(N) = \sum_{d|N} d^{k-1}$

Using $$\boxed{\zeta(k) = -\frac{(2\pi i)^k B_k}{2\,k!}} \quad (*)$$

where $$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1} \qquad B_0 = 0 \quad B_1 = \frac{-1}{2}$$ $$B_{2n+1} = 0 \quad n \geq 1$$

$$\left( B_2 = \frac{1}{2}, \quad B_4 = \frac{-1}{30}, \quad B_6 = 1/42 \right.$$

We see that $$E_k(\tau) := \frac{G_k(\tau)}{2\zeta(k)}$$

$$\boxed{E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n}$$

$$E_4(\tau) = 1 + 240q + 2160q^2 + \dots$$

$$E_6(\tau) = 1 - 504q - 16632q^2 + \dots$$

Exercise Prove $(*)$

$$\pi z \cot(\pi z) = \pi z i \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} = i\pi z \left( 1 + \frac{2}{e^{2\pi i z} - 1} \right)$$

$$= i\pi z + \frac{2\pi i z}{e^{2\pi i z} - 1} = i\pi z + \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i z)^k \qquad |z| < 1$$

On the other hand $$\pi z (\cot \pi z) = z \left( \frac{1}{z} + \sum_{d=1}^{\infty} \frac{1}{z+d} + \frac{1}{z-d} \right.$$

$$= 1 + 2z^2 \sum_{d=1}^{\infty} \frac{1}{z^2 - d^2}$$

$$\pi z \cot \pi z = 1 + 2z^2 \sum_{d=1}^{\infty} \frac{1}{d^2\left(\frac{z^2}{d^2} - 1\right)}$$

$$= 1 - 2z^2 \sum_{d=1}^{\infty} \frac{1}{d^2} \sum_{k=0}^{\infty} \left(\frac{z^2}{d^2}\right)^{2k}$$

$$= 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k} \qquad |z| < 1$$

Now the formula follows from the uniqueness of power series on their domain of convergence.

Before giving another proof of Prop 2.2 we note the following lemma

<u>Lemma 2.3</u> If $f : \mathbb{H} \to \mathbb{C}$ is holom. on $\mathbb{H}$ and $f(\tau + 1) = f(\tau)$

then $f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \tau}$

<u>Proof</u> $f(\tau + 1) = f(\tau) \implies f(\tau) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi i n x}$

$f$ hol $\iff \frac{\partial}{\partial \bar{\tau}} f = 0 \iff \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) f = 0.$

$$\implies \sum_{n \in \mathbb{Z}} \frac{\partial}{\partial \bar{z}} \left(a_n(y) e^{2\pi i n x}\right) = 0.$$

$$\implies \sum_{n \in \mathbb{Z}} \left(\frac{i}{2} a_n(y) . 2\pi n + \frac{i}{2} \frac{\partial}{\partial y} a_n(y)\right) e^{2\pi i n x} = 0$$

$$\implies \frac{\partial}{\partial y} a_n(y) = -a_n(y)(2\pi n) \implies a_n(y) = a_n e^{-2\pi n y}$$

Hence $f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \tau}$.

Note $a_n(y) = \int_{-1/2}^{1/2} f(x+iy)\, e^{-2\pi i n x}\, dx$

Hence $a_n(y)\, e^{2\pi n y} = \int_{-1/2}^{1/2} f(x+iy)\, e^{-2\pi i n z}\, dx$

$\Rightarrow a_n = a_n e^{-2\pi n y} \cdot e^{2\pi n y} = \int_{-1/2}^{1/2} f(z)\, e^{-2\pi i n z}\, dx$

2nd proof of Prop 2.2

We can prove Lipschitz's formula without using the cotangent series

Instead let $f(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k}$ $k>2$, $z \in \mathbb{H}$

$f(z)$ is holom. and periodic and $\lim_{z\to\infty} f(z) = 0$ Hence it has a Fourier expansion as above $\sum_{m=1}^{\infty} a_m e^{2\pi i m z}$

with $a_m = \int_0^1 f(z)\, e^{-2\pi i m z}\, dx$ for any $y = \mathrm{Im}\, z > 0$

$$a_m = \int_{z=0+iy}^{z=1+iy} f(z)\, e^{-2\pi i m z}\, dz$$

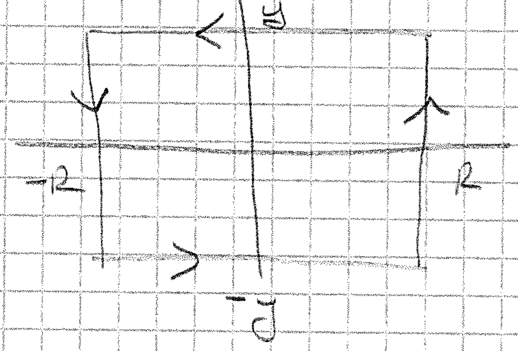
$$= \int_{0+iy}^{1+iy} \sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k}\, e^{-2\pi i m z}\, dz$$

Let $z = n + \mathfrak{z}$ to get

$$a_m = \sum_{n \in \mathbb{Z}} \int_{n+iy}^{n+1+iy} z^{-k} e^{-2\pi i m z} \, dz$$

$$= \int_{-\infty+iy}^{\infty+iy} z^{-k} e^{-2\pi i m z} \, dz.$$

To evaluate the last integral we use the residue theorem. Let $\gamma_R$ be the rectangular path




and consider

$$\int_{\gamma_R} z^{-k} e^{-2\pi i m z} dz$$

The integrand $z^{-k} e^{-2\pi i m z}$ is meromorphic with a singularity at $z = 0$.

$$-(2\pi i) \operatorname*{Res}_{z=0} \left( z^{-k} e^{2\pi i m z} \right) = \frac{(-2\pi i)^k}{(k-1)!} m^{k-1}$$

Hence by residue thm

$$\int_{\gamma_R} z^{-k} e^{-2\pi i m z} = \frac{(-2\pi i)^k}{(k-1)!} m^{k-1}$$

For $R \to \infty$ the upper horizontal part is $\displaystyle\int_{\infty + iy}^{-\infty + iy} z^{-k} e^{-2\pi i m z}\, dz = -a_m$

The other 3 integrals all go to zero as $R \to \infty$ and $y \to \infty$ since the integrand goes to zero.

This proves $\displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{(n+z)^k} = \sum_{m=1}^{\infty} \frac{(-2\pi i)^k}{(k-1)!} m^{k-1} e^{2\pi i m z}$

the Lipschitz's formula.

The rest of the proof is same as in proof 1.

3rd proof Uses the following thm

Thm (Poisson summation) Let $f \in L^1(\mathbb{R})$ such that its periodization $F(x) := \sum_{n\in\mathbb{Z}} f(x+n)$ converges abs. and uniformly on compact sets and is $\infty$ly often differentiable. Then

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} \hat{f}(m) e^{2\pi i m x}$$

where $\hat{f}(y) = \displaystyle\int_{-\infty}^{\infty} f(x) e^{-2\pi i x y}\, dx$ is its Fourier transform.

Apply PS to $f_y(x) = \dfrac{1}{(x+iy)^k}$ $\quad \tau = x+iy$, $y>0$

to calculate

$$\hat{f}(m) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x m}\, dx$$

$$= \int_{-\infty+iy}^{\infty+iy} \tau^{-k} e^{-2\pi i m \tau} \cdot e^{-2\pi m y}\, d\tau$$

$$= \begin{cases} a_m e^{-2\pi m y} & \text{with } a_m \text{ from prop 2 if } m>0 \\ 0 & m<0. \end{cases}$$

(Exercise)

<u>Rmk.</u> The Eisenstein series is a natural construction when one wants to construct a function which is invariant under the action of a group.

Recall from finite group theory if we have a linear action of a finite group $G$ on a vector space $V$, a natural way to construct a $G$-invariant vector in $V$ is to start with an arbitrary vector $v_0 \in V$ and form the sum $v := \sum_{g\in G} v_0|g$ (provided it is non-zero)

Here I write $v_0|g$ for the action of $g \in G$ on $v_0 \in V$

If the vector $v_0$ is invariant under a subgroup $G_0 \subset G$ then the vector $v_0|g$ depends only on the coset $G_0 g \in G_0\backslash G$ and we can form the smaller sum

$$v = \sum_{g \in G_0\backslash G} v_0|g$$

If $G$ is infinite the same method apply provided we have convergence.

If $v_0$ is fixed by an infinite s/gp $G_0$ then $G_0\backslash G$ is smaller and this improves chances of convergence.

In our setting $G = \Gamma = SL_2(\mathbb{Z})$ (or a s/gp)

We can take the constant function $1$

$$1|_k \gamma = (cz+d)^k \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $1|\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = 1$.

Hence $1$ is invariant under the s/gp

$$\Gamma_\infty = \pm\left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z}\right\}$$

The series $\displaystyle\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} 1|_k \gamma = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k}$

Check that a set of reps for $\Gamma_\infty \backslash \Gamma$

is $\{(c,d) \in \mathbb{Z}^2 \mid (c,d) = 1\} / (c,d)=(-c,-d)$

and the last series is $\frac{G_k(z)}{2\zeta(k)} = E_k(z)$.

---

The Eisenstein series have non-zero constant coefs. Recall that

<u>Defn</u> A cusp form of wt $k$ is a modular form $f$ of wt $k$ whose Fourier expansion has leading coef $a_0 = 0$. ie

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n$$

The set of cusp forms of wt $k$ for $\Gamma$ is denoted by $S_k(\Gamma)$

For $f \in M_k(\Gamma)$, if $f \in S_k(\Gamma)$ then

$$\lim_{\tau \to i\infty} f(\tau) = 0.$$

Since 2 modular (cusp) forms of wt $k_1$ and $k_2$ can be multiplied to give a modular (cusp) form of wt $k_1 + k_2$ we have a graded ring structure

$$M_{k_1}(\Gamma) M_{k_2}(\Gamma) \subset M_{k_1+k_2}(\Gamma) \qquad M_{k_1}(\Gamma) S_{k_2}(\Gamma) \subset M_{k_1+k_2}($$